

Partial classification of modules for Lie-algebra of diffeomorphisms of d -dimensional torus

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Abstract

We consider the Lie-algebra of the group of diffeomorphisms of a d -dimensional torus which is also known to be the algebra of derivations on a Laurent polynomial ring A in d commuting variables denoted by $\text{Der } A$. The universal central extension of $\text{Der } A$ for $d = 1$ is the so called Virasoro algebra. The connection between Virasoro algebra and physics is well known. See for example the book on Conformal Field Theory by Di Francesco, Mathieu and Senechal.

In this paper we classify $(A, \text{Der } A)$ modules which are irreducible and has finite dimensional weight spaces. Earlier Larsson constructed a large class of modules the so called tensor fields based on gl_d modules which are also A modules. We prove that they exhaust all $(A, \text{Der } A)$ irreducible modules.

Introduction

It is well known that the group of diffeomorphisms on a manifold is very important and shows up directly in many branches of physics (see for example Ref.22). We are particularly interested in d -dimensional torus. The case $d = 1$ is well studied by both mathematicians and physicists. The one dimensional central extension of the Lie-algebra of diffeomorphisms of the circle is well known object called Virasoro algebra. The representation theory of Virasoro algebra is studied in great detail. See Ref.13.

The Virasoro algebra acts on any (except when the level is negative of dual Coxeter number) highest weight module of the affine Lie algebra through the use of the famous Sugawara operators. It is well known that affine Lie algebras admit representation on the Fock space (see Ref.12) and hence admits a representation of the Virasoro algebra. This classical theory is what we originally want to generalize to d -dimensional torus.

The relation to physics is well established in the book on Conformal Field Theory by Di Francesco, Mathieu and Senechal. Ref.10. Specially the chapters 13 to 18 explains the connection between physics and the representation theory of Virasoro and affine Kac-Moody Lie-algebras. Several important papers on these aspects have been put in one volume by Goddard and Olive in Ref.8. The most fundamental paper in this direction is due to Belavin, Polyakov and Zamolodchikov in Ref.1.

The generalization of affine Lie algebra is the so called toroidal Lie-algebra. For the first time a large class of representations are constructed in Refs. 6 and 20 through the use of vertex operators generalizing the Fock space construction to the toroidal Lie-algebras. One significant difference for the toroidal Lie-algebra is that the universal center is infinite dimensional unlike in the affine case where it is one dimensional.

So the next natural question is to generalise Virasoro algebra and see whether the algebra acts on the Fock space. For that we first denote the Lie algebra of diffeomorphisms of d -dimensional torus by $\text{Der}A$ (it is known that $\text{Der}A$ is isomorphic to the derivations of Laurent polynomial ring A in d -variables). Here one should mention that several attempts has been made by physicists to give a Fock space representation to $\text{Der}A$ or to its extension (see Ref. 9). They all failed to produce any interesting results due to lack of proper definition of “normal ordering” among other things. At this juncture an interesting result has come out from Ref.21 which says that $\text{Der}A$ has no non-trivial central extension for $d \geq 2$.

Let us go back to the vertex construction of toroidal Lie-algebra of Ref. 6. Here operators are constructed for $\text{Der}A$ generalizing the Sugawara Construction. But the corresponding extension for $\text{Der}A$ is very wild (certainly non-central) and not tractable (see Ref.5). In the process an interesting abelian extension for $\text{Der}A$ has been created in Ref. 6 and the abelian part is exactly the center of the toroidal Lie algebra. So the semi-direct sum of the toroidal Lie algebra and $\text{Der}A$ with common extension has emerged as an interesting object which we will now define.

We will first define toroidal Lie algebra. Let \mathcal{G} be simple finite dimensional Lie algebra and let \langle, \rangle be a non-degenerate symmetric bilinear form on \mathcal{G} . Fix a positive integer d and let $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ be a Laurent polynomial ring in d -commuting variables.

Let Ω_A be the module of differentials which can be defined as vector space spanned by $t^r K_i$, $i = 1, \dots, d$ and $r \in \mathbb{Z}^n$. Let d_A be the subspace spanned by $\sum r_i t^r K_i$ and consider the toroidal Lie algebra $\mathcal{G} \otimes A \oplus \Omega_A/d_A$ with Lie-bracket.

$$[X \otimes t^r, Y \otimes t^s] = [X, Y] \otimes t^{r+s} + \langle X, Y \rangle \sum r_i t^{r+s} K_i$$

Ω_A/d_A is central.

Let $\text{Der}A$ be the derivation on A . For $u \in \mathbb{C}^d$ and $r \in \mathbb{Z}^d$ let $D(u, r) = \sum u_i t^r t_i \frac{d}{dt_i}$ where $u = (u_1, \dots, u_d)$ and $r = (r_1, \dots, r_d)$, $t^r = t_1^{r_1} t_2^{r_2} \dots t_d^{r_d} \in A$. Let $K(u, r) = \sum u_i t^r K_i$. Consider the Lie-algebra

$$\tau = \mathcal{G} \otimes A \oplus \Omega_A/d_A \oplus \text{Der}A$$

$$[D(u, r), D(v, s)] = D(w, r + s) - (u, s)(v, r)K(r, r + s)$$

where $w = (u, s)v - (v, r)u$ and (\cdot, \cdot) is the standard inner product in \mathbb{C}^d

$$[D(u, r), K(v, s)] = (u, s)K(v, r + s) + (u, v)K(r, r + s)$$

$$[K(u, r), K(v, s)] = 0$$

The first question is that can we construct a representation for τ from known methods. Several attempts have been made (Ref. 2, 3 and 6). Eventually in a remarkable paper Yuly Billig (Ref.24) has succeeded in constructing a class of modules for τ through the use of Vertex operator algebras (VOA). In the process Yuly Billig has used the $\text{Der}A$ modules constructed in Ref.16 and studied in Ref.4. One natural question is, does there exists more modules for $\text{Der}A$ so that we get a much larger class of models of τ .

In an interesting paper by Jiang and Meng (Ref.11) it is proved that classification of irreducible integrable modules of τ can be reduced to the classification of irreducible $(A, \text{Der}A)$ modules which the current paper settles. See also ref.7 for more precise results.

Let me explain the results of this paper in more detail. In Ref.16, Larsson constructed a large class of $\text{Der}A$ modules and some of them with finite dimensional weight spaces. In fact he constructed a functor from gl_d -modules to $\text{Der}A$ -modules. In Ref.4 the author proves that the image of an irreducible finite dimensional module is most often irreducible.

Further Larsson's $\text{Der}A$ modules are A modules too and they are always irreducible as $(A, \text{Der}A)$ -modules. Thus the purpose of the paper is to prove

the converse. So we prove in Theorem 1.9 that any $(A, \text{Der}A)$ module which is irreducible and has finite dimensional weight spaces has to come from Larsson's construction.

It will be certainly interesting to classify all irreducible $\text{Der}A$ modules with finite dimensional weight spaces. Now some kind of highest weight modules are constructed in Ref.2 and 3 (They are not A modules). We will also note that $GL(d, \mathbb{Z})$ acts as automorphisms on $\text{Der}A$ and so we can twist a $\text{Der}A$ module by $GL(d, \mathbb{Z})$. (Larsson's modules are closed under $GL(d, \mathbb{Z})$ action). So it will be interesting to prove that any irreducible $\text{Der}A$ module with finite dimensional weight spaces is either an highest weight module or a Larsson's module upto a twist of $GL(d, \mathbb{Z})$ action. The problem is completely solved for $d=1$ by Mathieu in Ref.19

Our results in Ref.6 have been reinterpreted in the language of physics by Larsson in Refs. 17 and 18. Larsson is also first to talk about non-central extensions is Ref. 15. $\text{Der}A$ has also been studied in Ref.23.

1. Section

(1.1) Throughout this paper we fix a positive integer $d \geq 2$ and a Laurent polynomial ring $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ in d commuting variables. Let \mathbb{C}^d be d copies of complex field \mathbb{C} . Let e_1, \dots, e_d be the standard basis of \mathbb{C}^d and let $(,)$ be the standard form on \mathbb{C}^d such that $(e_i, e_j) = \delta_{ij}$.

(1.2) Let $\Gamma = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_d$. Throughout this paper we use m, n, r and s to denote elements of Γ . For $r = \sum r_i e_i \in \Gamma$ let $t^r = t_1^{r_1} t_2^{r_2} \dots t_d^{r_d} \in A$ and let $D^i(r) = t^r t_i \frac{d}{dt_i}$ be a derivation on A . Let $\text{Der}A$ be the Lie-algebra of derivations of A . It is easy to verify that $D^i(r), 1 \leq i \leq d, r \in \Gamma$ is a basis of $\text{Der}A$. For $u = \sum u_i e_i \in \mathbb{C}^d$ let $D(u, r) = \sum u_i D^i(r)$. Then $\text{Der}A$ has the following Lie structure:

(1.3)

$$[D(u, r), D(v, s)] = D(w, r + s)$$

where $w = (u, s)v - (v, r)u$, $r, s \in \Gamma$ and $u, v \in \mathbb{C}^d$. Let h be the subspace spanned by $D^i(0)$, $1 \leq i \leq d$, which is a maximal abelian subalgebra of $\text{Der}A$.

(1.4) Note that $D(u, r)t^s = (u, s)t^{r+s}$. Thus $A \oplus \text{Der}A$ is a Lie-algebra by extending the Lie structure in the following way

$$[t^r, t^s] = 0$$

$$[D(u, r), t^m] = (u, m)t^{r+m}.$$

Let $\tilde{h} = \mathbb{C} \oplus h$ which is an abelian subalgebra of $A \oplus \text{Der}A$.

The purpose of this paper is to study $A \oplus \text{Der}A$ modules which are weight modules for \tilde{h} with finite dimensional weight spaces and to classify such modules with some natural conditions.

We first recall $\text{Der}A$ modules which are constructed and studied in References 16 and 4.

(1.5) Let gl_d be the Lie-algebra of $d \times d$ matrices with entries in \mathbb{C} . Let E_{ij} be the elementary matrix with (i, j) th entry 1 and zero elsewhere. Then it is well known that gl_d is spanned by E_{ij} , $1 \leq i, j \leq d$ with the following Lie-bracket

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}.$$

Let $gl_d = sl_d \oplus \mathbb{C}I$ where sl_d is a Lie-subalgebra of trace zero matrices and I is the identity matrix. Let $V(\psi)$ be the irreducible finite dimensional module for sl_d where ψ is a dominant integral weight. Let I act by scalar b on $V(\psi)$ and denote the resultant gl_d module by $V(\psi, b)$. Let $\alpha \in \mathbb{C}^d$ and we will make $F^\alpha(\psi, b) := V(\psi, b) \otimes A$ a $\text{Der}A$ module. First denote $v \otimes t^m$ by $v(m)$ for v in $V(\psi, b)$ and m in Γ .

(1.6) Definition (Ref. 16)

$$D(u, r) \cdot v(m) = (u, m + \alpha)v(m + r) + \left(\sum_{i,j} u_i r_j E_{ji} v\right)(m + r)$$

where $m, r \in \Gamma, u \in \mathbb{C}^d, v \in V(\psi, b)$. We will now recall the following

(1.7) Theorem (Theorem (1.9) and Proposition (5.1) of Ref.4).

(1) $F^\alpha(\psi, b)$ is irreducible as $\text{Der}A$ module if $(\psi, b) \neq (\delta_k, k), (0, b), 1 \leq k \leq d - 1$ where δ_k is the k th fundamental weight of sl_d .

(2) $F^\alpha(0, b)$ is irreducible as $\text{Der}A$ module unless $\alpha \in \Gamma$ and $b \in \{0, d\}$.

In all other cases $F^\alpha(\psi, b)$ is reducible and the submodule structure has been worked out in Proposition (5.1) and Theorem (5.5) of Ref.4.

Recall that A is associative algebra with unit and $F^\alpha(\psi, b)$ is a A module by defining

$$t^m \cdot v(r) = v(m + r)$$

for $m, r \in \Gamma$ and $v \in V(\psi, b)$. Further it is easy to see that $F^\alpha(\psi, b)$ is a $A \oplus \text{Der}A$ module.

(1.8) Proposition $F^\alpha(\psi, b)$ is irreducible as $A \oplus \text{Der}A$ module.

Proof First note that $F^\alpha(\psi, b)$ is a weight module with respect to \tilde{h} and the weight spaces are $V(\psi) \otimes t^m$. Suppose W is a non-zero $A \oplus \text{Der}A$ submodule of $F^\alpha(\psi, b)$. As submodule of a weight module is a weight module, W is a weight module. From the action of A it is clear that $v(m) \in W$ implies $v(s) \in W$ for all $s \in \Gamma$. Thus $W = W_1 \otimes A$ for some $W_1 \subseteq V(\psi, b)$. Now choose $u = e_i, r = e_j$ and consider

$$D(u, r)v(m) = (u, m + \alpha)v(m + r) + (E_{ji}v)(m + r).$$

It now follows from the above remarks that W_1 is gl_d -invariant. Since $V(\psi)$ is irreducible and W_1 is non-zero, it follows that $W_1 = V(\psi)$ and hence $W = F^\alpha(\psi, b)$.

The purpose of this paper is to prove converse of the above proposition. In other words we classify $A \oplus \text{Der}A$ modules with certain natural properties.

(1.9) Theorem Let V be irreducible module for $A \oplus \text{Der}A$ which is also a weight module for \tilde{h} with finite dimensional weight spaces. We further assume the following:

- (1) V is a A -module as associative algebra and the Lie-module structure of A comes from associative algebra.
- (2) $1.v = v, \forall v$ in V .

Then $V \cong F^\alpha(\psi, b)$ (for some α, ψ, b) as $A \oplus \text{Der}A$ -module.

We need to develop several lemmas to prove the theorem which will be done in Section 2. The final proof will be given in Section 3.

2. Section

First we need to change some notation. We treat A as group algebra over Γ . For that let $k(r)$ be a symbol for $r \in \Gamma$. Let A be the linear span of $k(r), r \in \Gamma$ with multiplication defined as $k(r) \cdot k(s) = k(r + s)$.

Let U be the universal enveloping algebra of $A \oplus \text{Der}A$. Let L be the two sided ideal of U generated by $k(r)k(s) - k(r + s)$ and $k(0) - 1$.

Throughout this section the module V is as in Theorem (1.9). Since V is a A module, L acts trivially and hence V is a U/L -module. Let $V = \oplus_{r \in \Gamma} V_r$ be the weight space decomposition and $V_r = \{v \in V \mid D(u, 0)v = (u, r + \alpha)v, \forall u \text{ in } \mathbb{C}^d\}$. Such a uniform α in \mathbb{C}^d exists as V is irreducible. In fact take any weight space where h acts as linear function which can be taken as $u \mapsto (u, \alpha)$ for some $\alpha \in \mathbb{C}^d$. Because of irreducibility the action of h on the rest of the

spaces is easily computed. Further each V_r is a \tilde{h} -module as 1 in A acts as one on the entire module.

(2.1) Let $U_1 = U/L$ and let $T(u, r) = k(-r)D(u, r) - D(u, 0)$ as an element of U_1 for $u \in \mathbb{C}^d$ and $r \in \Gamma$. Let T be the subspace spanned by $T(u, r)$ for all u and r .

(2.2) Proposition

- (1) $[T(v, s), T(u, r)] = (u, s)T(v, s) - (v, r)T(u, r) + T(w, r + s)$
where $w = (v, r)u - (u, s)v$ and hence T is a Lie-subalgebra.
- (2) $[D(v, 0), T(u, r)] = 0$
- (3) $V = \oplus V_r$ be weight space decomposition. Then each V_r is T -invariant
- (4) Each V_r is T -irreducible
- (5) $V_r \cong V_s$ as T -module.

Proof

$$\begin{aligned}
(2) \quad [D(v, 0), k(-r)D(u, r) - D(u, 0)] \\
&= [D(v, 0), k(-r)D(u, r)] \\
&= [D(v, 0), k(-r)]D(u, r) \\
&+ k(-r)[D(v, 0), D(u, r)] \\
&= -(v, r)k(-r)D(u, r) \\
&+ (v, r)k(-r)D(u, r) \\
&= 0.
\end{aligned}$$

(1) From (2) it follows that

$$[T(v, s), T(u, r)] = [k(-s)D(v, s), k(-r)D(u, r)]$$

$$\begin{aligned}
&= [k(-s), k(-r)D(u, r)]D(v, s) + k(-s)[D(v, s), k(-r)D(u, r)] \\
&= [k(-s), k(-r)]D(u, r)D(v, s) \\
&+ k(-r)[k(-s), D(u, r)]D(v, s) \\
&+ k(-s)[D(v, s), k(-r)]D(u, r) \\
&+ k(-s)k(-r)[D(v, s), D(u, r)] \\
&= (u, s)k(-s)D(v, s) \\
&\quad - (v, r)k(-r)D(u, r) \\
&+ k(-s-r)D(w, r+s) \text{ where } w = (v, r)u - (u, s)v \\
&= (u, s)T(v, s) \\
&\quad - (v, r)T(u, r) \\
&+ T(w, r+s) \\
&\text{where } w = (v, r)u - (u, s)v
\end{aligned}$$

- (3) From (2) it follows that T commutes with h and hence V_r is a T -module.
- (4) Let $U = \bigoplus_{r \in \Gamma} U_r$ where $U_r = \{v \in U \mid [D(u, 0), v] = (u, r)v \text{ for } u \in \mathbb{C}^d\}$. Since V is $A \oplus \text{Der} A$ irreducible for v, w in V_r there exists X in U_0 such that $Xv = w$. This is due to weight reasons. Now $X = \sum a_i X_i$ where each X_i is of the form $k(-r)D(u_0, r_1) \cdots D(u_k, r_k)$ where $\sum r_i = r$. We are using the fact that L acts trivially on V . Now using the fact that $k(-s)D(u, r) = D(u, r)k(s) - (u, s)D(u, r)$ and the fact that $k(r)k(s) = k(r+s)$ we see that each X_i is linear combination of elements of the form

$$k(-r_1)D(u_1, r_1)k(-r_2)D(u_2, r_2) \cdots k(-r_k)D(u_k, r_k).$$

This proves $X \in U(T)$, the universal enveloping algebra of T . Hence V_r is T irreducible.

- (5) First note that $k(s-r)V_r \subseteq V_s$. Repeating the same we see that

$$V_r = k(r-s)k(s-r)V_r \subseteq k(r-s)V_s \subseteq V_r.$$

Thus $V_r = k(r - s)V_s$. Define $f : V_r \rightarrow V_s$ by $f(v) = k(s - r)v$ which is clearly injective and surjective. Now

$$\begin{aligned} f(T(u, k)v) &= k(s - r)T(u, k)v \\ &= T(u, k)k(s - r)v \\ &= T(u, k)f(v) \end{aligned}$$

Thus f is a T -homomorphism. This proves (5).

(2.3) Notation For any integer $k > 0, r, m_1, \dots, m_k \in \Gamma$ define

$$\begin{aligned} T_k(u, r, m_1, \dots, m_k) &= T(u, r) - \sum_i T(u, r + m_i) + \sum_{i < j} T(u, r + m_i + m_j) \cdots \\ &(-1)^j \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} T(u, r + m_{i_1} + \dots + m_{i_j}) \cdots (-1)^k T(u, r + m_1 + m_2 + \dots + m_k). \end{aligned}$$

Let I_k be the linear span of $T_k(u, r, m_1 \cdots m_k), u \in \mathbb{C}^d$ for all $r, m_1, m_2 \cdots m_k \in \Gamma$.

(2.4) Lemma (1) $T_k(u, r, m_1, \dots, m_k) = T_k(u, r, m_{\sigma(1)}, \dots, m_{\sigma(k)})$ for any permutation σ on k -letters.

(2) $T_k(u, r, m_1, \dots, m_k) = T_{k-1}(u, r, m_1, \dots, m_{k-1}) - T_{k-1}(u, r + m_k, m_1, \dots, m_{k-1})$

(3) I_k is an ideal of T .

(4) $I_k \subseteq I_{k-1}$ for $k \geq 2$.

(5) $[I_k, I_\ell] \subseteq I_{k+\ell-1}$ for $k, \ell \geq 1$.

Proof (1) Follows from definition

(2) Collect all terms where m_k does not occur in the sum of T_k and that can be seen to be equal to $T_{k-1}(u, r, m_1, \dots, m_{k-1})$. Sum of the rest of the terms can be seen to equal to $-T_{k-1}(u, r + m_k, m_1, \dots, m_{k-1})$. This is because every term contains m_k

$$(3) [T(v, s), T_k(u, r, m_1, \dots, m_k)] =$$

$$\begin{aligned} & (u, s) \sum_{\ell=0}^k \binom{k}{\ell} (-1)^\ell k(-s) D(v, s) \\ & - (v, r) T_k(u, r, m_1, \dots, m_k) \\ & + \sum (v, m_i) T_{k-1}(u, r + m_i, m_1, \dots, \widehat{m_i}, \dots, m_k) \\ & + (v, r) T_k(u, r + s, m_1, \dots, m_k) \\ & - \sum (v, m_i) T_{k-1}(u, r + s + m_i, m_1, \dots, \widehat{m_i}, \dots, m_k) \\ & - (u, s) T_k(v, r + s, m_1, \dots, m_k). \end{aligned}$$

By applying Proposition 2.2 (1) write $[T(v, s), T_k(u, r, m_1, \dots, m_k)] = A_1 + A_2 + A_3$. It is easy to see that A_1 is the first term of the above formula. Now in A_2 look for the terms where (v, m_i) occurs and that can be seen as a component of the third term of the above formula. Now in A_3 the terms where no m_i occurs is equal to the second term of the above formula. The rest of the formula can be seen in a similar way. This proves the claim. Now note that the first term in the claim is zero. Clearly 2nd, 4th and 6th terms are in I_k . Now 3rd and 5th term is equal to

$$\begin{aligned} & \sum (v, m_i) T_{k-1}(u, r + m_i, m_1, \dots, \widehat{m_i}, \dots, m_k) \\ & - \sum (v, m_i) T_{k-1}(u, r + m_i + s, m_1, \dots, \widehat{m_i}, \dots, m_k) \\ & = \sum (v, m_i) T_k(u, r + m_i, m_1, \dots, \widehat{m_i}, \dots, m_k, s) \end{aligned}$$

(by Lemma 2.4 (2)).

(4) Follows from (2).

(5)

$$\begin{aligned} & [T_\ell(v, s, n_1, \dots, n_\ell), T_k(u, r, m_1, \dots, m_k)] \\ & \sum_{t=0}^{\ell} \sum_{i_1 < i_2 < \dots < i_t} (u, s + n_{i_1} + \dots + n_{i_t}). \end{aligned}$$

$$\begin{aligned}
& \sum_{b=0}^k \binom{k}{b} (-1)^b T(v, s + n_{i_1} + \cdots + n_{i_t}) \\
& - \sum_{t=0}^k \sum_{j_1 < \cdots < j_t} (v, r + m_{j_1} + \cdots + m_{j_t}). \\
& \sum_{b=0}^{\ell} \binom{\ell}{b} (-1)^b T(u, r + m_{i_1} + \cdots + m_{i_t}) \\
& + (v, r) T_{k+\ell}(u, r + s, m_1, \cdots, m_k, n_1, \cdots, n_{\ell}) \\
& - (u, s) T_{k+\ell}(v, r + s, m_1, \cdots, m_k, n_1, \cdots, n_{\ell}) \\
& - \sum (v, m_i) T_{k+\ell-1}(u, r + s + m_i, m_1, \cdots, \widehat{m_i}, \cdots, m_k, n_1, \cdots, n_{\ell}) \\
& + \sum (u, n_j) T_{k+\ell-1}(v, r + s + n_j, m_1, \cdots, m_k, n_1, \cdots, \widehat{n_j}, \cdots, n_{\ell})
\end{aligned}$$

The above formula can be deduced as in (3) from Proposition 2.2 (1). Now note that the first two terms are zero as

$$\sum_{b=0}^k \binom{k}{b} (-1)^b = 0 = \sum_{b=0}^{\ell} \binom{\ell}{b} (-1)^b.$$

The rest of the four terms are in $I_{k+\ell-1}$ and this proves (5).

(2.5) Lemma For $u \in \mathbb{C}^d, 0 \neq m_i \in \Gamma, s \in \Gamma$.

(1) $T_k(u, s, m_1, \cdots, m_k) \notin I_{k+1}$ for $k \geq 1$.

(2) $T_k(u, s, m_1, \cdots, m_k)$

$$+ T_k(u, s, n, m_2, \cdots, m_k) = T_k(u, s, m_1 + n, m_2, \cdots, m_k) + I_{k+1}$$

(3) $T_k(u, s, -m_1, m_2, \cdots, m_k)$

$$= -T_k(u, s - m_1, m_1, m_2, \cdots, m_k)$$

Proof To prove the Lemma, we first interpret T_k 's as certain polynomials in $A = \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. We fix a non-zero u in \mathbb{C}^d . Let k be a positive integer and let $m_1, m_2, \dots, m_k \in \Gamma$. Let $P_k(m_1, \dots, m_k) = \prod_{1 \leq i \leq k} (1 - t^{m_i})$. Recall $t^{m_i} = t_1^{(m_i)_1} \dots t_d^{(m_i)_d}$. Let J_k be the ideal in A generated by $P_k(m_1, \dots, m_k)$ for all non-zero m_i 's $\in \Gamma$. Then clearly $J_{k+1} \subseteq J_k$. It is easy to see that $T_k(u, r, m_1, \dots, m_k)$ can be identified with polynomial $t^r P_k(m_1, \dots, m_k)$. Recall that u is fixed.

Thus it is sufficient to prove that,

Claim 1 $P_k(m_1, \dots, m_k) \notin J_{k+1}$. Suppose

$$(*) \quad P_k(m_1, \dots, m_k) = \sum f_\ell P_{k+1}(n_{\ell_1}, n_{\ell_2}, \dots, n_{\ell_{k+1}})$$

where $f_\ell \in A$. Let $Dt_i = t_i \frac{d}{dt_i}$. Now consider $D_{t_{i_1}} \dots D_{t_{i_k}} P_{k+1}(n_1, \dots, n_{k+1})$ and evaluate at $(t_1, \dots, t_d) = (1, \dots, 1)$. This can be seen to be zero as after differentiating P_{k+1} , k times, each component has at least one factor $(t^{n_i} - 1)$. We will now prove that there exists i_1, \dots, i_k such that

Claim 2 $D_{t_{i_1}} \dots D_{t_{i_k}} P_k(m_1, \dots, m_k) |_{t=(1, \dots, 1)}$ is non-zero. Thus $*$ can not hold. This prove the claim 1. Now choose $\ell, 1 \leq \ell \leq d$ such that $S = \{i \mid (m_i)_\ell \neq 0\}$ is non-empty. Let $\#S = p$ and let $i_1, \dots, i_p \in S$. Consider

$$(D_{t_\ell})^p P_k(m_1, \dots, m_k) = \mu \prod_{i \in S} (m_i)_\ell \prod_{j \notin S} (1 - t^{m_j}) t^{m_{i_1} + m_{i_2} + \dots + m_{i_p}} + J_{k-s+1}$$

which is not too difficult to see. Where μ is a non-negative integer. Repeating the process finitely many times (choosing different index $\ell^1 \neq \ell$). We see that there exists i_1, \dots, i_k such that $D_{t_{i_1}} D_{t_{i_2}} \dots D_{t_{i_k}} P_k(m_1, \dots, m_k) = \lambda t^{m_1 + \dots + m_k} + J_1$ where λ is non-zero integer. Now evaluating at $t = (1, \dots, 1)$ we see that claim 2 is true.

To see (2) first note that

$$(1 - t^m)(1 - t^n) + (1 - t^{m+n}) = (1 - t^m) + (1 - t^n).$$

Thus

$$\begin{aligned}
& t^s \prod_{i=2}^k (1 - t^{m_i})(1 - t^{m_1}) + t^s \prod_{i=2}^k (1 - t^{m_i})(1 - t^n) \\
&= t^s \prod_{i=2}^k (1 - t^{m_i})(1 - t^n)(1 - t^{m_1}) \\
& \quad + t^s \prod_{i=2}^k (1 - t^{m_i})(1 - t^{m_1+n}).
\end{aligned}$$

This proves (2).

(3) is easy to check.

(2.6) Lemma

$$(1) \quad \text{Dim}(I_k/I_{k+1}) \leq d^{k+1}, k \geq 1$$

$$(2). \quad T = I_1$$

In particular I_k is a co-finite ideal in T .

Proof First note that from Lemma 2.4(2) we have

$$T_k(u, r, m_1, \dots, m_k) = T_k(u, s, m_1 \dots m_k) \text{ mod } I_{k+1}$$

for all $r, s \in \Gamma$. Further

$$-T_k(u, 0, m_1, \dots, m_k) = T_k(u, 0, -m_1, m_2 \dots m_k) \text{ mod } I_{k+1}$$

which follows from above and Lemma 2.5 (3). Now from additive property of Lemma 2.5(2) it follows that I_k/I_{k+1} is spanned by $T_k(u, 0, e_{i_1}, \dots, e_{i_k})$ where e_1, \dots, e_d is the standard basis. Thus (1) follows. (2) follows from definitions. Now it is easy to conclude that I_k is a co-finite ideal for each k .

3. Section

We will explain the plan of the proof of Theorem (1.9). First we will prove that $T/I_2 \cong gl_d(\mathbb{C})$. Then we will prove that if $I_k, k \geq 2$ is zero on a finite dimensional irreducible module V of T then I_2 is zero on V . Thus V is a module for $T/I_2 \cong gl_d(\mathbb{C})$. Further we prove that any co-finite ideal J of T contains I_k for large k . Thus any irreducible finite dimensional module V of T is actually a module for T/I_2 . From this it will be easy to conclude Theorem 1.9 which will be explained at the end of the section.

(3.1) Proposition $T/I_2 \cong gl_d(\mathbb{C})$.

Proof First recall that $F^\alpha(\psi, b)$ is a $A \oplus \text{Der}A$ -module and each weight space $V(\psi) \otimes t^m$ is a T -module. It is easy to verify that I_2 acts trivially on $V(\psi) \otimes t^m$. Now note that $T(e_i, e_j)v(m) = E_{ji}v(m) \neq 0$ for some ψ . From this we conclude that $T(e_i, e_j)$ is non-zero in T/I_2 . Now it is easy to see that $T(u, s) + T(u, r) = T(u, r + s) \text{ mod } I_2$ and hence $T(e_i, e_j)$ spans T/I_2 . Define $\pi : T/I_2 \rightarrow gl_d(\mathbb{C})$.

$$\pi(T(e_i, e_j)) = E_{ji}$$

Consider

$$\begin{aligned} X &= [T(e_i, e_j), T(e_k, e_\ell)] \\ &= [k(-e_j)D(e_i, e_j), k(-e_\ell)D(e_k, e_\ell)] \\ &= -\delta_{i\ell}k(-e_\ell)D(e_k, e_\ell) \\ &\quad + \delta_{kj}k(-e_j)D(e_i, e_j) \\ &\quad - \delta_{kj}k(-e_\ell - e_j)D(e_i, e_\ell + e_j) \\ &\quad + \delta_{i\ell}k(-e_\ell - e_j)D(e_k, e_\ell + e_j). \end{aligned}$$

Follows from Proposition (2.2). Note that the following is true in T/I_2 .

$$k(-e_\ell - e_j)D(e_s, e_\ell + e_j) = k(-e_\ell)D(e_s, e_\ell)$$

$$+k(-e_j)D(e_s, e_j) - D(e_s, 0)$$

for $s = i, k$. Thus $X = -\delta_{kj}(k(-e_\ell)D(e_i, e_\ell) - D(e_i, 0))$

$$\begin{aligned} & +\delta_{i\ell}(k(-e_j)D(e_k, e_j) - D(e_k, 0)) \\ & = -\delta_{kj}T(e_i, e_\ell) + \delta_{i\ell}T(e_k, e_j). \end{aligned}$$

Thus π defines a surjective homomorphism. As $T(u, 0)$ is zero it follows that $T(e_i, e_j)$ span T/I_2 which proves $\dim (T/I_2) \leq d^2$. Thus π defines an isomorphism.

(3.2) Lemma (Yuly Billig) Suppose \mathcal{G} is a Lie- algebra over \mathbb{C} and J is an ideal with spanning set $J_\alpha, \alpha \in B$. Suppose there exists an element I in \mathcal{G} such that $[I, J_\alpha] = \lambda J_\alpha, \lambda \neq 0$ for all $\alpha \in B$. then J acts trivially on any irreducible finite dimensional module V of \mathcal{G} .

Proof Since the base field is complex numbers and V is finite dimensional, I has eigen vectors. Let $\lambda_1, \dots, \lambda_k$ be all the eigen values of I on V . Choose λ_i such that $\lambda + \lambda_i$ is not an eigenvalue. Let v be eigenvector with eigenvalue λ_i for I . Consider $IJ_\alpha v = J_\alpha Iv + [I, J_\alpha]v = (\lambda_i + \lambda)J_\alpha v$. This proves $J_\alpha v = 0 \forall \alpha \in B$. Let $W = \{w \in V \mid J_\alpha w = 0 \forall \alpha \in B\}$. Since J is an ideal, it is easy to see that W is a \mathcal{G} -module. But $W \neq 0$. Since V is irreducible $W = V$ which proves that J acts trivially on V .

(3.3) Proposition Suppose V is irreducible finite dimensional module for T such that I_{k+1} acts trivially on V . Then I_2 acts trivially on V .

Proof From the proof of Lemma 2.4(3) we have

$$\begin{aligned} & [T(v, s), T_k(u, r, m_1, \dots, m_k)] = -(v, r)T_{k+1}(u, r, m_1, \dots, m_k, s) \\ & -(u, s)T_k(v, r + s, m_1, \dots, m_k) + \sum (v, m_i)T_k(u, r + m_i, m_1, \dots, \widehat{m_i}, \dots, m_k, s). \end{aligned}$$

Let $I = \sum T(e_i, e_i)$ and note that I is actually identity element in $T/I_2 \cong g\ell_d(\mathbb{C})$. Thus I is non-zero on T/I_{k+1} for $k \geq 1$.

Claim

$$[I, T_k(u, r, m_1, \dots, m_k)] = (k-1)T_k(u, r, m_1, \dots, m_k).$$

Consider

$$\begin{aligned} \left[\sum_j T(e_j, e_j), T_k(u, r, m_1, \dots, m_k) \right] &= - \sum_j u_j T_k(e_j, r + e_j, m_1, \dots, m_k) \\ &\quad + \sum_{i,j} (m_i)_j T_k(u, r + m_i, m_1, \dots, \widehat{m_i}, \dots, m_k, e_j). \end{aligned}$$

Now we use Lemma 2.5(2) and the following facts.

- (1) T_k is linear in u
- (2) $T_k(u, r + m, m_1, \dots, m_k) = T_k(u, r, m_1, \dots, m_k) \bmod I_{k+1}$ (by lemma 2.4(2)).
- (3) I_{k+1} is zero on V .

From that we conclude that

$$[I, T_k(u, r, m_1, \dots, m_k)] = -T_k(u, r, m_1, \dots, m_k) + kT_k(u, r, m_1, \dots, m_k)$$

which proves the claim. Now we can use Lemma (3.2) for the ideal I_k . Thus I_k is zero on V . Repeating this argument we conclude that I_2 acts trivially on V . This argument breaks down for $k = 1$ as we cannot apply the Lemma 3.2.

(3.4) Proposition Any co-finite ideal J of T contains I_k for large k .

Proof Claim $J \cap I_k$ is co-finite in T for all k . For that consider $\varphi : T \mapsto T/J \oplus T/I_k$

$$v \mapsto (v, v).$$

Clearly $\ker \varphi = J \cap I_k$ and $T/J \cap I_k$ is a subalgebra of finite dimensional Lie-algebra $T/J \oplus T/I_k$. This proves the claim.

Consider

$I_k/J \cap I_k \xrightarrow{\varphi_1} I_{k-1}/J \cap I_k \xrightarrow{\varphi_2} I_{k-1}/J \cap I_{k-1}$ where φ_1 is injective and φ_2 is surjective. Let $\rho_k = \varphi_2 \circ \varphi_1$. Thus $\rho_k : I_k/I_k \cap J \rightarrow I_{k-1}/J \cap I_{k-1}$. Clearly ρ_k is injective. Let $t_n = \dim I_n/I_n \cap J$ and note that $t_{n+1} \leq t_n$. Thus $\{t_n\}_{n \in \mathbb{Z}^+}$ is a decreasing sequence of non-negative integers. Therefore $t_k = s$ for some s and for large $k > N$.

First we note the following two statements for a fixed i .

(1) For $\ell \neq i$

$$[T(e_\ell, -e_\ell), T_k(e_i, 0, e_{j_1}, \dots, e_{j_k})] = -k_\ell T_k(e_i, 0, e_{j_1}, \dots, e_{j_k})$$

where k_ℓ is the number of e_ℓ that occur in $T_k(e_i, 0, e_{j_1}, \dots, e_{j_k})$.

(2) Suppose the ideal J contains $\sum a_{m,I} T_m(e_i, 0, e_{j_1}, \dots, e_{j_m})$ where the number of e_i 's that occur in $T_m(e_i, 0, e_{j_1}, \dots, e_{j_m})$ is same for all m where $I = \{j_1, \dots, j_k\}$. Then J contains $T_m(e_i, 0, e_{j_1}, \dots, e_{j_m})$ for $m \ni a_{m,I} \neq 0$.

(1) Follows from the proof of lemma 2.4(3). (2) follows from (1). We will prove the Proposition assuming $d \geq 3$ to avoid some computations. For a fixed i , consider the following set

$$S = \{T_k(e_i, 0, e_{j_1}, \dots, e_{j_k}) \mid j_\ell \neq i \text{ for all } \ell\}.$$

Now choose $k \ni \#S > s$ and $k > N$. Thus S is linearly dependent mod $I_k \cap J$. Thus there exists non-zero scalars $a_I (I = \{j_1, \dots, j_k\})$ such that

$$X = \sum a_I T_k(e_i, 0, e_{j_1}, \dots, e_{j_k}) \in J.$$

Now using (2) we conclude that

$$T_k(e_i, 0, e_{j_1}, \dots, e_{j_k}) \in J \text{ for some } I.$$

For $m, n \neq i$ consider

$$[T(e_m, e_n), T_k(e_i, 0, e_{j_1}, \dots, e_{j_k})] = \ell \delta_{j_i m} T_k(e_i, e_m, e_{j_1}, \dots, \widehat{e}_{j_i}, \dots, e_{j_k}, e_n) \in J.$$

$$\text{Now } T_k(e_i, e_m, e_{j_1}, \dots, \widehat{e}_{j_i}, \dots, e_{j_k}, e_n) = T_k(e_i, 0, e_{j_1}, \dots, \widehat{e}_{j_i}, \dots, e_{j_k}, e_n)$$

$$-T_{k+1}(e_i, 0, e_{j_1}, \dots, e_{j_i}, \dots, e_{j_k}, e_n).$$

Now by (2) it follows that

$$T_k(e_i, 0, e_{j_1}, \dots, \widehat{e}_{j_i}, \dots, e_{j_k}, e_n) \in J.$$

Now repeating this process we see that

$$(*) \quad T_k(e_i, 0, e_{j_1}, \dots, e_{j_k}) \in J$$

for all possible indices j_1, \dots, j_k which are all different from i . Applying $T(e_i, e_i)$ to the above vector to conclude

$$T_k(e_i, e_i, e_{j_1}, \dots, e_{j_k}) \in J$$

$$(**) \quad T_k(e_i, 0, e_{j_1}, \dots, e_{j_k}) - T_k(e_i, e_i, e_{j_1}, \dots, e_{j_k})$$

$$= -T_{k+1}(e_i, 0, e_{j_1}, \dots, e_{j_k}, e_i) \in J.$$

Fix $j \neq i$. Replacing k by $k+1$, consider the following vector which is in J by (*).

$$[T(e_j, e_i), T_{k+1}(e_i, 0, e_{j_1}, \dots, e_{j_{k+1}})] = P \delta_{j j_\ell} T_{k+1}(e_i, e_j, e_{j_1}, \dots, \widehat{e}_{j_\ell}, \dots, e_{j_{k+1}}, e_i)$$

$$-T_{k+1}(e_j, e_i, e_{j_1}, \dots, e_{j_{k+1}}).$$

Now $T_{k+1}(e_i, e_j, e_{j_1}, \dots, \widehat{e}_{j_\ell}, \dots, e_{j_{k+1}}, e_i)$

$$= T_{k+1}(e_i, 0, e_{j_1}, \dots, \widehat{e}_{j_\ell}, \dots, e_{j_{k+1}}, e_i) - T_{k+2}(e_i, 0, e_{j_1}, \dots, e_j, \dots, e_{j_{k+1}}, e_i).$$

Now by $(**)$ both vectors are in J . Thus we conclude that

$$T_{k+1}(e_j, e_i, e_{j_1}, \dots, e_{j_{k+1}}) \in J.$$

Now by (2) we see that

$$T_{k+1}(e_j, 0, e_{j_1}, \dots, e_{j_{k+1}}) \in J.$$

This is true for all possible indices j_1, \dots, j_{k+1} which are all different from i .

Now applying $T(e_{j_\ell}, e_i)$ for $j_\ell \neq j$ we see that

$$T_{k+1}(e_j, e_{j_\ell}, e_{j_1}, \dots, \widehat{e}_{j_\ell}, \dots, e_{j_{k+1}}, e_i) \in J.$$

Now by (2) we see that

$$T_{k+1}(e_j, 0, e_{j_1}, \dots, \widehat{e}_{j_\ell}, \dots, e_{j_{k+1}}, e_i) \in J.$$

Repeating this process we see that $T_{k+1}(e_j, 0, e_{\ell_1}, \dots, e_{\ell_{k+1}}) \in J$ for all possible $\ell_1, \dots, \ell_{k+1}$.

Now using the technique in the proof of Lemma 2.5 we see that

$$T_{k+1}(e_j, 0, m_1, \dots, m_{k+1}) \in J \text{ for all } m_i \in \Gamma$$

Now replasing $k+1$ by $k+2$ we see that

$$T_{k+2}(e_j, 0, m_1, \dots, m_{k+2}) = T_{k+1}(e_j, 0, m_1, \dots, m_{k+1}) - T_{k+1}(e_j, m_{k+2}, m_1, \dots, m_{k+1}) \in J.$$

Then it follows that

$$T_{k+1}(e_j, m_{k+2}, m_1, \dots, m_{k+1}) \in J.$$

Strictly speaking we have it for non-negative coefficients. But the other cases can be handled similarly. This proves $I_{k+1} \subseteq J$ and the Proposition. Further $s = 0$.

Proof of Theorem (1.9) . Let V be a module as in Theorem. Let $V = \bigoplus_{r \in \Gamma} V_r$ be the weight space decomposition where

$$V_r = \{v \in V \mid D(u, 0)v = (u, r + \alpha)v, \forall u \in \mathbb{C}^d\}.$$

We know that $V_r \cong V_s$ as T -modules from Proposition 2.2(5). This with Proposition (3.3), Proposition (3.4) combined with the fact that some co-finite ideal of T acts trivially on V_r tells us that all V_r 's are isomorphic to some $V(\psi, b)$ as gl_d -modules. Note that the isomorphism between V_r 's is given by $k(r)$ (from proof of proposition 2.2(5)). Thus if we let $V_r = V(\psi, b) \otimes t^r$ we see that $k(r)v(s) = v(s + r)$ for v in $V(\psi, b)$.

Now consider $T(u, r)$ in T/I_2 and note that it is linear in both variables.

Thus

$$\begin{aligned} T(u, r)v(s) &= \sum_{i,j} u_i r_j T(e_i, e_j)v(s) \\ &= \sum_{i,j} u_i r_j E_{ji}v(s) \end{aligned}$$

Therefore $k(-r)D(u, r)v(s) = D(u, 0)v(s) + (\sum u_i v_j E_{ji}v)(s)$

$$= (u, s + \alpha)v(s) + (\sum u_i r_j E_{ji}v)(s).$$

Multiply both sides by $k(r)$ we get

$$\begin{aligned} D(u, r)v(s) &= (u, s + \alpha)v(s + r) \\ &+ (\sum_{ij} u_i r_j E_{ji}v)(s + r). \end{aligned}$$

This completes the proof of the theorem.

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